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Note

Ramsey's Theorem for Sums, Products, and Arithmetic Progressions

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Given a finite partition $\{A_i\}_{i=1}^r$ of the set N of positive integers, van der Waerden's theorem [4] says that we may choose i such that A_i contains arbitrarily long arithmetic progressions; the finite sum theorem [1] says that we may choose j such that A_j contains $FS(B)$ for some infinite $B \subseteq N$ (where $FS(B)$ is the set of all finite sums of distinct members of B); and the finite product theorem (a trivial corollary of the finite sum theorem) says that we may choose k such that A_k contains $FP(C)$ for some infinite $C \subseteq N$ (where $FP(C)$ is the set of all finite products of distinct members of C). It is known [2, Theorem 2.6] that we can choose $j = k$. We show here that we can in fact choose $i = j = k$.

We work with the semigroup $(\beta N, \cdot)$, where βN is the Stone–Čech compactification of N and $\cdot$ is the left-continuous extension of ordinary multiplication to βN which has N contained in (in fact equal to) the center of $(\beta N, \cdot)$. The points of βN are the ultrafilters on N. In order to keep this note short, we refer the reader to [3] for an elementary description of $(\beta N, \cdot)$.

DEFINITION. (a) $\bar{\Gamma} = \{p \in \beta N$: for each $A \in p$ there is some infinite $B \subseteq N$ with $FS(B) \subseteq A\}$.

(b) $\mathscr{A}\mathscr{P} = \{p \in \beta N$: each $A \in p$ contains arbitrarily long arithmetic progressions$\}$.

LEMMA 1. *$\bar{\Gamma}$ is a closed right ideal of $(\beta N, \cdot)$.*

Proof. This is what was proved in [2, Lemma 2.5]. ∎

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LEMMA 2. *$\mathscr{A}\mathscr{P}$ is a closed left ideal of $(\beta N, \cdot)$.*

Proof. That $\mathscr{A}\mathscr{P} \neq \varnothing$ follows from van der Waerden's theorem and Theorem 6.7 of [3] (with $\mathscr{G} = \{A \subseteq N$: A contains arbitrarily long arithmetic progressions$\}$). To see that $\mathscr{A}\mathscr{P}$ is closed, let $p \in \beta N \backslash \mathscr{A}\mathscr{P}$ and pick $A \in p$ which does not contain arbitrarily long arithmetic progressions. Then $\bar{A} = \{q \in \beta N: A \in q\}$ is a neighborhood of p which misses $\mathscr{A}\mathscr{P}$. Finally, to see that $\mathscr{A}\mathscr{P}$ is a left ideal, let $p \in \mathscr{A}\mathscr{P}$, let $q \in \beta N$, and let $A \in q \cdot p$. We need to show that A has arbitrarily long arithmetic progressions so let $l \in N$. Let $B = \{x \in N: A/x \in q\}$. Then $B \in p$ so pick $a, d \in N$ such that $\{a + td: t \in \{0, 1, 2,..., l\}\} \subseteq B$. Then $\bigcap_{t=0}^{l} A/(a+td) \in q$ so pick $b \in \bigcap_{t=0}^{l} A/(a+td)$. Then $\{ba + t(bd): t \in \{0, 1, 2,..., l\}\} \subseteq A$ as required. ∎

We remark that in fact $\mathscr{A}\mathscr{P}$ is a (2-sided) ideal of $(\beta N, \cdot)$ and of $(\beta N, +)$.

THEOREM. *Let $N = \bigcup_{i=1}^{r} A_i$ (where $r \in N$). Then there exists $i \in \{1, 2,..., r\}$ such that A_i contains arbitrarily long arithmetic progressions and contains $FS(B) \cup FP(C)$ for some infinite B and C contained in N.*

Proof. By Lemmas 1 and 2 $\bar{\Gamma} \cap \mathscr{A}\mathscr{P}$ is a nonempty compact left-topological semigroup. Hence (see, e.g., [3, Lemma 8.1]) there is an idempotent p in $\bar{\Gamma} \cap \mathscr{A}\mathscr{P}$. Pick $i \in \{1, 2,..., r\}$ such that $A_i \in p$. Since $p \in \bar{\Gamma}$, A_i contains $FS(B)$ for some infinite $B \subseteq N$. Since $p \in \mathscr{A}\mathscr{P}$, A_i contains arbitrarily long arithmetic progressions. Since $p = p \cdot p$ we have by [3, Theorem 8.6] (due to Galvin) that A_i contains $FP(C)$ for some infinite $C \subseteq N$. ∎

We remark that the conclusions of [3, Theorem 9.7] also hold for the cell A_i chosen above.

REFERENCES

1. N. HINDMAN, Finite sums from sequences within cells of a partition of N, *J. Combin. Theory Ser. A* **17** (1974), 1–11.
2. N. HINDMAN, Partitions and sums and products of integers, *Trans. Amer. Math. Soc.* **247** (1979), 227–245.
3. N. HINDMAN, "Ultrafilters and Combinatorial Number Theory," Lecture Notes in Math. No. 751, pp. 119–184, Springer-Verlag, New York/Berlin, 1979.
4. B. VAN DER WAERDEN, Beweis einer Baudetschen Vermutung, *Nieuw Arch. Wisk. (3)* **19** (1927), 212–216.